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Research paper

ARCHITECTURAL DESIGN TOOL BASED ON KOENIGS MESHES

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Abstract

Koenigs meshes should be more explored in the field of architectural design due to their various fabrication efficiency factors for doubly curved grid-shells. Beside being planar quad meshes (PQ-meshes) they have additional property of duality and are closely linked to graphical statics: their diagonals and those of their dual mesh represent form and force diagrams in equilibrium under a normal unit load. These meshes also fulfill essential geometric criteria such as planar panels suitable for glass cladding. Special subclasses—like discrete isothermic surfaces and discrete minimal surfaces—offer additional advantages, including quad blocks with planar lateral sides (zero geometric torsion) and constant edge offsets. The latter allows for simplified fabrication using straight, discrete strips. Koenigs meshes remain underutilized in architectural practice. This research develops design morphology techniques and parametric tools that preserve their geometric properties, and introduces computational methods for constructing and exploring designs with these networks. The study offers new design strategies of dual meshes and presents practical workflows for implementing Koenigs nets in architectural applications.

Key words: *discrete Koenigs surfaces, discrete isothermic surfaces, discrete minimal surfaces, architectural geometry, PQ meshes*

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1. INTRODUCTION

Koenigs meshes are discrete analog of Koenings patches with specific fabrication and structural advantages, making them highly relevant in grid-shell architecture. They represent planar quad meshes (PQ-meshes) that also support duality, making them highly relevant for construction applications. Their structural significance arises from their direct relationship with their dual meshes, particularly in graphical statics, where the diagonals (of the quads) of a Koenigs mesh and its dual represent the *form* and *force* diagram of one another. This property enables them to reach static equilibrium under an external normal unit load [1, 2]. Additionally, they satisfy essential geometric criteria for design, including planar quad faces, which facilitate prefabrication of their panels on the flat platform or glass cladding. Furthermore, the intersecting diagonals points of adjacent faces produce planar panels, presenting a promising solution as a second option for glass covering, when the primary construction is derived from the diagonals.

Within the scope of Koenigs meshes class is their subclass of *discrete Isothermic Surfaces (or IS-meshes)* which further contain *discrete Minimal Surfaces (or MS-meshes)*. They bring additional fabrication advantages to the construction elements such as: „square-like“ proportionality of the panels, zero geometric torsion (face sides without twisting) [1], and in the case of discrete minimal surfaces also: *constant edges offset* blocks whose planar lateral sides unfold to a straight band. These properties are very rare in the doubly curved grid-shells, but enable simpler fabrication techniques with the possibility of material savings during production since the elements can be produced by cutting and folding parallel stripes of material [3].

While their methods of construction and properties are known among mathematicians and researchers within discrete differential geometry [2], [4–6], they have not been widely explored for applications in the design of doubly curved grid-shells. For the first time, this research applies transformation principles that preserve the geometric properties of Koenigs meshes to broaden the range of design variations within pre-rationalization design approch. The main contribution of this paper is developing the tools and algorithms for constructing these meshes is based on defined steps, driven from geometry theory, within parametric modeling.

2. GEOMETRY DEFINITIONS

In this section we will talk about geometry definitions and main properties of the Koenigs meshes as well as their subclasses of IS-meshes and MS-meshes. The theory from this section will be used for development of the design tool for K-meshes grid-shells.

2.1. Koenigs meshes (K-meshes)

In this paper, by mesh we mean a quad mesh that is, a 2-dimensional array of points in the standard 3-dimensional Euclidean space. When the faces of the mesh are planar we call this a *Planar Quad mesh* (or PQ-mesh), note that all the meshes studied here are PQ. Next, a Koenigs mesh (K-mesh) is a PQ-mesh that admits a dual K-mesh. To understand this, we begin by defining the dual of a quad. So let (A,B,C,D) be a starting quad, then, its dual transform, denoted *(A*,B*,C*,D*),* is a quad such that all the edges are parallel to the initial quad while opposite diagonals are parallel (Figure 1).

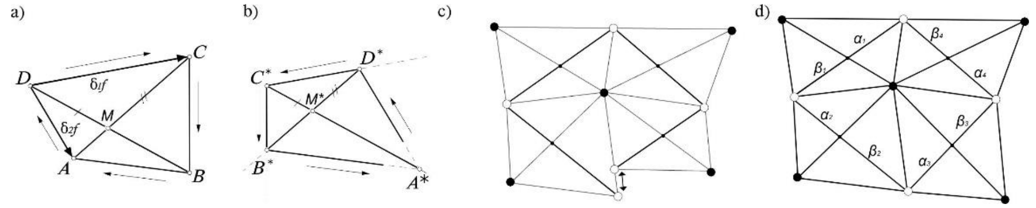

Figure 1. a) Elementary quad b) Its dual transformation; c) Dual of four arbitrary quads d) Closure condition on K-mesh

The dual transform of a quad defined above is the central operation that we will use here. We can then say, that a K-mesh is PQ-mesh that admits a dual mesh, which is another mesh where the corresponding elementary quads are related by the dual transform described. This is intuitive enough; however, it does not tell us how to construct a K-mesh. A natural way to explore, is to start by a PQ-mesh and apply the dual transform to each of its quads. The result set of quads will not necessarily close together properly into a neat mesh (Figure 1c). For this to happen we have to impose another condition on the initial PQ-mesh, that we call the closure condition. To this end, given a point with its four adjacent planar faces, we define the distances (on the diagonals) denoted by $\alpha_1, \alpha_2, \ldots$ as seen in Figure 1d. Now the closure condition is that at every point the relation (1) is satisfied. Intuitively, it can be said that the diamond net around every point has to be closed. For further details and proof, see [2].

It then follows, that a K-mesh can be defined as a mesh that satisfies:

- Planarity of faces (being PQ-mesh)
- Closure condition (admitting a dual mesh).

$$\frac{\beta_1}{\alpha_1} \cdot \frac{\beta_2}{\alpha_2} \cdot \frac{\beta_3}{\alpha_3} \cdot \frac{\beta_4}{\alpha_4} = 1 \tag{1}$$

Interesting and unique geometric property of K-meshes beyond planar elementary quadrilaterals is that the intersection points of the diagonals that belong to four adjacent quadrilaterals are coplanar [5]. Additionally, Koenigs meshes are projectively invariant, meaning their two fundamental properties (planarity and closure condition), remain unchanged under projective transformations (PT). In intuitive design exploration, the PT can be seen as a transformation that stretches the mesh in different directions while preserving face planarity and keeping the diamond nets around each point closed. This characteristic will be the basis of their construction in this paper. Now, PT (in dimension 3) can be interpreted as an operation on the standard Euclidean space given by the mapping:

$$(x', y', z') = \left(\frac{a_1x + a_2y + a_3z + a_4}{d_1x + d_2y + d_3z + d_4}, \frac{b_1x + b_2y + b_3z + b_4}{d_1x + d_2y + d_3z + d_4}, \frac{c_1x + c_2y + c_3z + c_4}{d_1x + d_2y + d_3z + d_4}\right) \tag{2}$$

where the 4x4 matrix whose rows are given by the input parameters a_1,a_2,a_3,a_4; b_1,b_2,b_3,b_4; c_1,c_2,c_3,c_4, and d_1,d_2,d_3,d_4 has a non-zero determinant. These are used to generate shape variations. Particulary this equasion is used to transform the coordinates of starting points (x, y, z) of the mesh into a new set of points of the gained mesh with coordinates (x', y', z').

2.2. Discrete Isothermic surfaces (IS-meshes)

IS-meshes are in fact a specialization of K-meshes where we require more constrains on the meshes. More precisely, we first require that quads of our PQ-mesh to be Cyclic, that means that all the vertices are co-circular (lie on the same circle), we call such a mesh a circular mesh. Further, we require the circular mesh to satisfy what is called the Cross- Ratios condition. This means that the lengths of the four edges of adjacent quads must satisfy a specific relationship. To understand that, we recall that the cross-ratio of a cyclic quad (A, B, C, D) is defined by the relations $\left(\frac{|AB|}{|BC|}\right).\left(\frac{|CD|}{|DA|}\right)$, where | | means the length of the edge. Given a point of a circular mesh and its four adjacent faces, let q_1, q_2, q_3, q_4 be the cross- ratios of these faces, the Cross-Ratios condition is given by the relation:

$$q_1 \cdot q_3 = q_2 \cdot q_4 \quad (3)$$

An IS-mesh is thus defined as a mesh satisfying:

- Circularity (being circular mesh)
- Cross-Ratio condition [5].

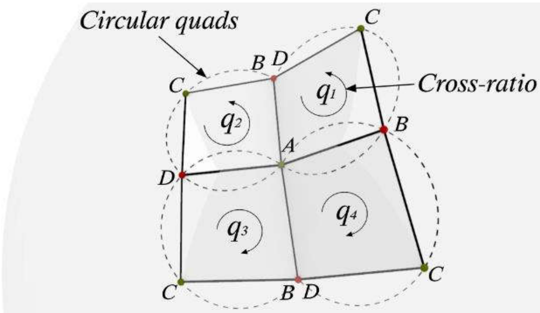

Figure 2. Cross-Ratios of four adjacent faces at a point

Being in particular a K-mesh, an IS-mesh naturally admits *a dual mesh, using the dual* transform of its elementary quads as seen above. However, in the case of IS-mesh this dual transform will be called a Christoffel dual transform, reflecting its origin in smooth classical setting. Furthermore, the two fundamental *properties of IS*-meshes are not invariant (they change) under PT however, they are invariant under Mobius transformations (MT), which will be explained later on. Finally, it is known that being a circular mesh is equivalent to being a Vertex-Offset mesh (VO-mesh) [1], [7], [8]. This means that their neibouring discrete normals are coplanar which is a significant fabrication advantage for modular faces and nodal construction elements.

2.3. Discrete minimal surfaces

MS-meshes belong to a subclass of IS-meshes that is called s-IS-meshes. These are defined as collection of vertices one black (b-vert) and two types of whites (c-vert) and (s-vert). Moreover, they satisfy the following conditions:

- the four b-vertices surrounding a c-vertex are co-circular (lie in same circle)
- the four b-vertices surrounding a s-vertex are co-spherical (lie in same sphere)
- these circles and spheres intersect orthogonally (at the b-vertices).

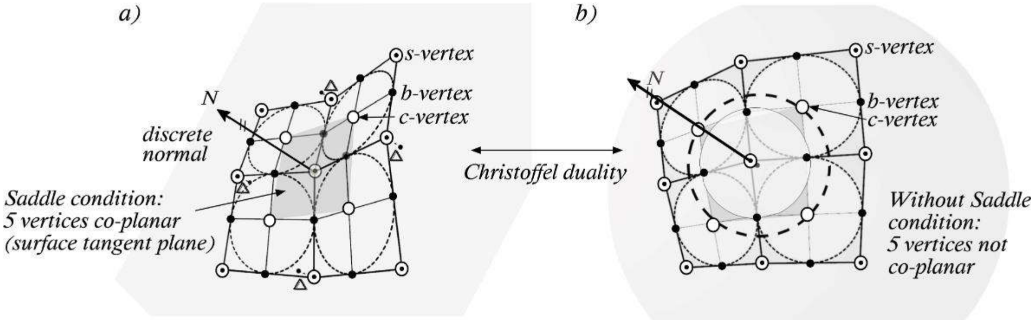

Figure 3. Properties of discrete minimal surface and Koebe

We will call the above conditions the s-IS-mesh condition, and thus a mesh satisfying this will be called an s-IS-mesh, and we index only the s-vertices as our mesh points. Furthermore, it can be observed that the four s-vertices surrounding a c-vertex are co-planar, and their face has inscribed circle (centered at the c-vertex and passing through the b-vertices), as seen in Figure 3a) [6]. Now, if we require that the s-vertices, are co-planar with their surrounding c-vertices (before they were co-spherical), then we observe that the four adjacent faces at every s-vertex take the saddle-like configuration. We call this the Saddle condition.

We can thus define an MS-mesh as a mesh that satisfies the two conditions:

- s-IS-mesh condition (being s-IS-mesh)
- Saddle condition (every white vertex is co-planar with its white neighbors).

The s-IS-mesh condition is invariant (does not change) under MT while the Saddle condition is not. Now, as the others, MS-mesh admits a Christoffel dual mesh, which is known to be an s-IS-mesh on the sphere—that is, a Koebe mesh. We will use exactly this property to construct MS-mesh. Then lets see what is a Koebe mesh.

2.1.1. Koebe polyhedral and edge offset – EO mesh

Note that the plane obtained at the s-vertex with its surrounding c-vertices is the tangent plane to the MS-mesh and the normal to this plane is the discrete normal at the s-vertex (Figure 3a). When transformed into its dual, it can be seen that its edges are tangent to the unit sphere at the transforms of the b-vertices, while the transforms of the s-vertices lie outside the sphere. Such a mesh is called a Koebe mesh [9]. It follows that each face (defined by the transformed of the s-vertices) has an inscribed circle which is precisely the intersection of the face in question and the sphere (Figure 4). So far, we explained that the Christoffel dual of an MS-mesh is a Koebe mesh, however we want to do the opposite construction. Namely, constructing a Koebe mesh then getting an MS-mesh by taking the Christoffel dual. To this end, we recall that a Koebe mesh can be directly constructed from the two orthogonal Circle Packings (CP) on the unit sphere, which in turn give us MS-mesh by applying Christoffel dual transform [4]. In this paper, we construct the two orthogonal CP by constructing them in the plane (the (u, v) domain) then send them to the unit sphere using the inverse of the stereographic projection (which is known to preserve circles), denoted by N_o and given by:

$$N_o(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2u}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right) \quad (4)$$

Secondly, we can pre-compose N_o with I, the inversion in a circle of radius R, or in general with ψ, an MT (in the Complex plane), given respectively by:

$$I(u,v) = \left(\frac{R^2}{u^2+v^2}\right)(u,v) \quad with\ R\ non-zero \tag{5}$$

$$\psi(T) = \frac{aT+b}{cT+d} \text{ with } T = u + iv \text{ and } a,b,c,d \text{ such that } ad-bc \text{ non}-\text{zero} \tag{6}$$

The resulting conformal map from these pre-compositions $(N_o \circ I\,,\, N_o \circ \psi)$ will be denoted N (parametrization sending the circle packing pattern on the sphere). Furthermore, we can also post-compose these with an MT (in 3-dim.). The MT can be formulated as a combination of scaling, translation and inversion. The third one can be intuitively described as a motion of turning the mesh "inside out" through a sphere or a plane. It keeps the geometry of circles and spheres as well as angles. That is why it can be implemented on IS-meshes without changing their properties. Also, it can be implemented on MS-meshes, where only the first condition is kept, so we get s-IS-mesh from MS.

Despite being somewhat less intuitive a MT can be more conveniently defined as a 5x5 matrix μ. whose generators are the matrices denoted by A, B, C, D. These are themselves parameterized respectively by a, b, c, d varying between -1 and 1 (cf. digital tool). The reason for the 5 dimensions, comes from the rather abstract realization of the Mobius transformations as Lorentz transformations preserving a light-cone in Minkowski space of dimension 5. The data of the transform is sent to and from this using stereographic projection in dimension 3. For more details about its construction see [10].

In order to construct a proper circle packing by using these parametrizations it is important to divide the parametric domain into squares which can have inscribed circles. This can be made by having the same u, v domain and dividing them by same number $ndu = ndv$, or all values can be varied but by keeping the same step value $(u_n - u_{n+1} = v_n - v_{n+1})$. After using the chosen combination of mapping we have the circles on the sphere. Since here we are looking at the quad subdivision, the four planes P_1, P_2, P_3, P_4 of adjacent circles should intersect into one point A (Figure 4b). The intersection of the each of neighboring two planes gives us the tangent edges of the sphere. If repeated for the all four adjacent circles, we get all the vertices and edges of the Koebe mesh.

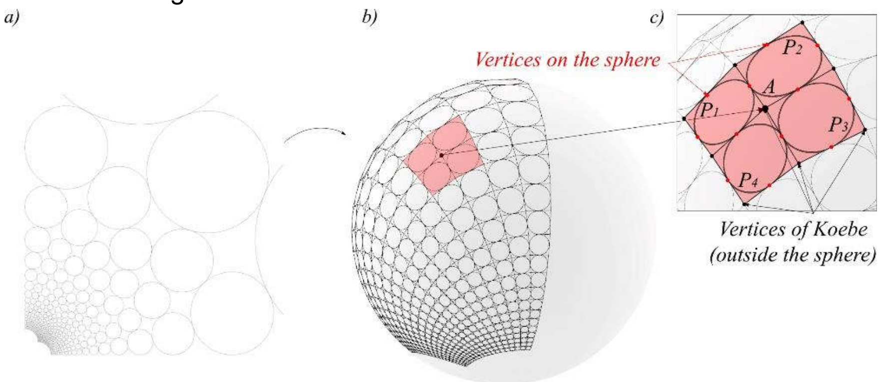

Figure 4. Construction of Koebe polyhedra by using inversion in plane and inverse stereographic [11]

Now that we showed how we can easily construct a Koebe mesh, we go back to its relation with MS-meshes. As mentioned above, a Koebe mesh is the discrete Gauss map of an MS-mesh (which is also its Christoffel dual), this means that this MS-mesh is Edge Offset (EO) mesh, i.e. admitting an offset mesh with constant edge to edge distance [1].

Similarly, as the Koebe mesh is as well an EO-mesh (since it is its own discrete Gauss map). In practical terms, if denote the Koebe mesh by M1 and its EO-mesh by M2, the corresponding edges are parallel (since at constant distance) and that if connect the corresponding b-vertices they all meet at the center (origin) of the unit sphere, as seen in Figure 5. Note that the b-vertices on M_1 all lie on the unit sphere and are the points of tangency between it and the Koebe mesh.

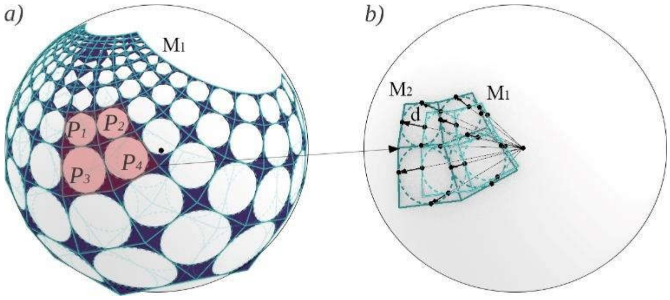

Figure 5. a) Koebe polyhedra M_1; b) Four adjacent faces with tangent edges on the sphere M_1 with its offset mesh M_2 [11]

2.1.2. Edge offset straight unfolding property

It turns out that the MS-meshes (and negatively curved meshes in general) posses an extra property among the EO-meshes (i.e. meshes having their discrete Gauss maps as Koebe meshes). That is, the EO-block B obtained by connecting the EO-mesh with it offset, will have not only planar lateral sides (zero geometric torsion) but also the *unfold into a straight strip*, as seen in Figure 6. This is a direct consequence of the Saddle condition. To understand this intuitively, take a point on the MS-mesh and its four adjacent faces, and the Face Offset (FO) blocks (in red) from the Face normals constructed at these faces. It is then not so difficult to see that the saddle condition of the MS-mesh will force the common faces of four adjacent EO-blocks B to always lie between the faces of the FO-blocks. In particular, they (B) overlap in one direction and drift apart in the other, as seen in Figure 6 (middle). This in turn will induce a specific angle θ for each EO-block by which the planes of its lateral faces will rotate in and out in an alternating manner, as seen in Figure 6 (right). This alternation of rotation is the cause of the lateral sides unfolding into a straight strip, as seen in Figure 6 (right). This clearly provides us with a significant fabrication advantages, as the EO-blocks can be cut from straight pieces reducing cutting time and material loss.

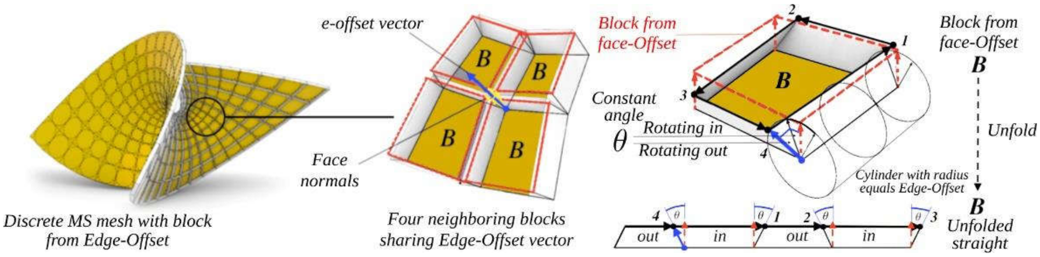

Figure 6. Example of the discrete MS mesh with the construction of its edge offset and proof of its straight unfolded sides [11]

3. DESIGN WORKFLOW AND MORPHOLOGY

This section focuses on constructing a different Koenigs nets and its subclass by using appropriate transformations based on the geometry theory shown in Section 2 for expanding the design space of grid-shells using Koenigs meshes. The goal is to showcase the developed tool for parametric design of these meshes as well as give instructions on how to use the separate tools to generate one of the three mentioned mesh types. The setup of the construction is done by starting from the most rigid level and loosening the geometric properties with transformations, getting the other two.

Three construction starting points are presented for designing grid-shells with dual meshes (Figure 7). The design process always begins with a square grid in the u, v plane, ensuring that the divisions remain square. After this, Mobius transformations are used to deform the grid in plane and inverse stereographic projection is applied to create two orthogonal circle packings on the sphere (giving rise to a Koebe mesh). In third case the conformal mapping of tangent circles is done directly on the sphere. All of the steps are divided into different Grasshopper (GH) components with their input parameters which makes possible to combine them easily and intuitively without the necessity of geometry theory from the Section 2 (Figure 7).

Dual meshes are divided into three levels: Level I represent MS-mesh, obtained by applying Christoffel duality to the Koebe meshes. Level II is achieved by applying a Möbius transformation to *Level I*, which relaxes the minimal surface condition but retains the isothermic mesh properties. Level III can be achieved by applying a projective transformation to Level II, losing the circular mesh property but maintaining the Koenigs property. Alternatively, a projective transformation can be applied directly to Level I, resulting in Level III. Despite differences in geometric properties, all mesh levels are Koenigs, sharing the advantages of dual meshes.

It is also important to emphasize that, except in the case of Level I, in the other two levels (II and III), it is possible to alternate the application of Möbius or projective transformations with Christoffel duality in order to obtain different forms of Level II or III (indicated in yellow and green in Figure 7). This property enables the generation of an unlimited number of different forms of dual meshes. In the continuation, the application of the strategies for constructing dual networks across the three levels is presented in following subsections.

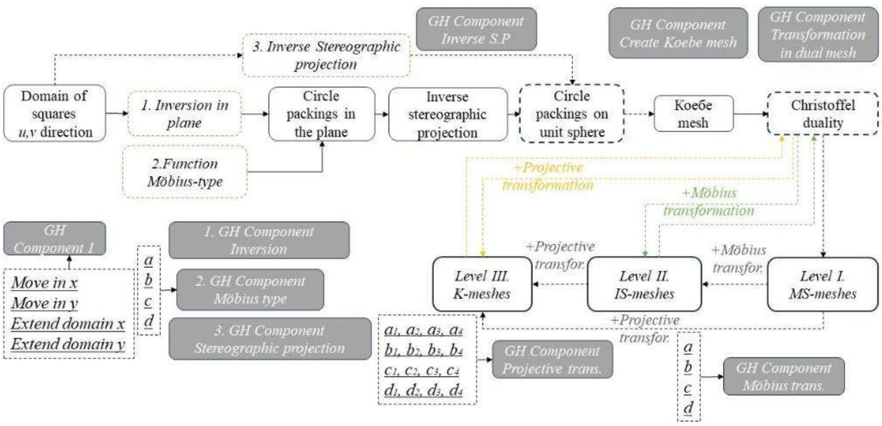

Figure 7. Strategies for constructing Koenigs meshes within three defined levels with their GH components and input parameters

3.1 Procedure for Constructing Discrete Minimal Surfaces – Level I

The first subsection outlines the procedure for constructing Level I dual meshes – discrete minimal surfaces. Since this represents the level with most geometrical constrains, the fabrication advantages include not only planar panels, but also nearly orthogonal edges, square-like quad proportions, and straight unfoldable strips of all box elements. These properties remain unchanged regardless of the design steps taken using GH components. To construct them parametrically, the three conformal mapping cases are considered (Figure 7), with corresponding design procedures. The first case involves applying an inversion to a 20x20 square grid with the origin at $(0,0,0)$ (Figure 8a), resulting in a deformed circle packing (Figure 8b).

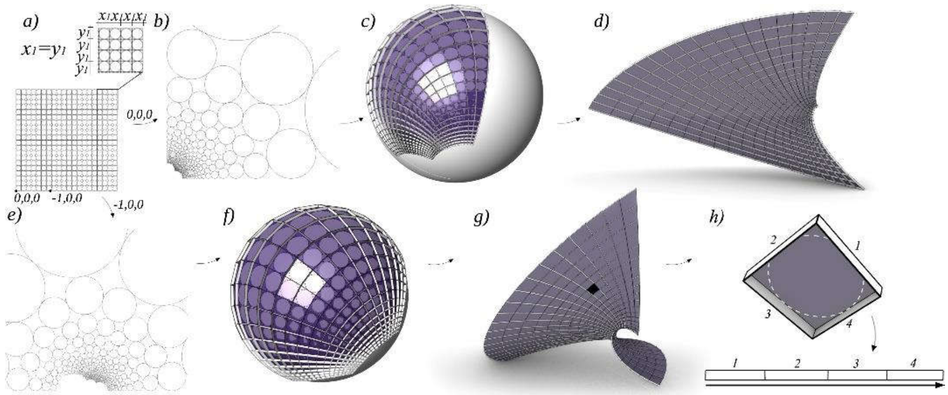

Figure 8. Level I mesh: Start with a square grid (u, v), apply circle inversion, then use inverse stereographic projection to create a Koebe mesh. Apply Christoffel duality to obtain the MS-mesh.

Next, the inverse stereographic projection is applied sending the circle packing in the (u, v) domain to a circle packing in the sphere. The Koebe mesh is then constructed using these circles and the offset, defined by the normals at the tangent points of the mesh edges (Figure 8c). All Level I networks thus have an edge offset. The Christoffel duality procedure is then applied to create an MS-mesh of Enneper type [4] – Level I (Figure 8d). The discrete normals are used to construct its edge offset mesh (EO-mesh) of the Enneper mesh, which give rise to quad blocks, that unfold straight. This holds true for all the presented Level I cases (Figure 8h, Figure 9h, Figure 10h).

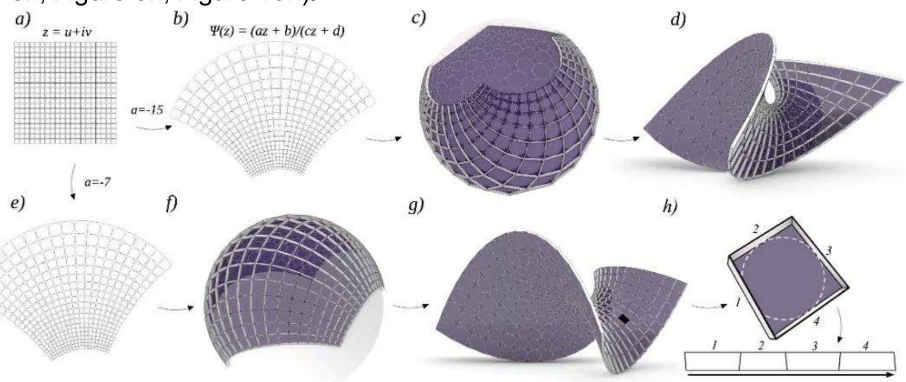

Figure 9. Level I: Möbius map and inverse stereographic projection yielding discrete minimal surfaces via Christoffel duality.

This MS-mesh of Enneper type construction itself is not new, but this work focuses on developing tools and input parameters for experimenting with shell forms. By adjusting the position of the initial u, v plane relative to the inversion circle's center, the grid resolution, and the inversion circle's radius, different mesh forms can be generated. For example, shifting the starting plane from (0,0,0) to (-1,0,0) alters the circle packing (Figure 8d), Koebe mesh (Figure 8f), and final Enneper mesh form (Figure 8g). Increasing the grid rows stretches the mesh, revealing a larger mesh domain. These steps are implemented as components in the GH program, allowing users to modify the grid-shell form at each stage by changing the components input parameters.

The second example of a discrete minimal surface is derived from a Möbius-type conformal mapping Ψ in the plane. Starting with the division of the u,v plane into the standard initial circle packing (Figure 9a), the mapping is applied, creating a deformed circle packing in u, v plane (Figure 9b). Using inverse stereographic projection, the circles are mapped onto the circles in the sphere, which give rise to in the Koebe mesh (Figure 9c). Applying Christoffel duality then generates the discrete minimal surface (Figure 9d). The Möbius transformation generator parameters $a, b, c,$ and d control the mesh's form, with changes shown in Figure 9d, e, and f.

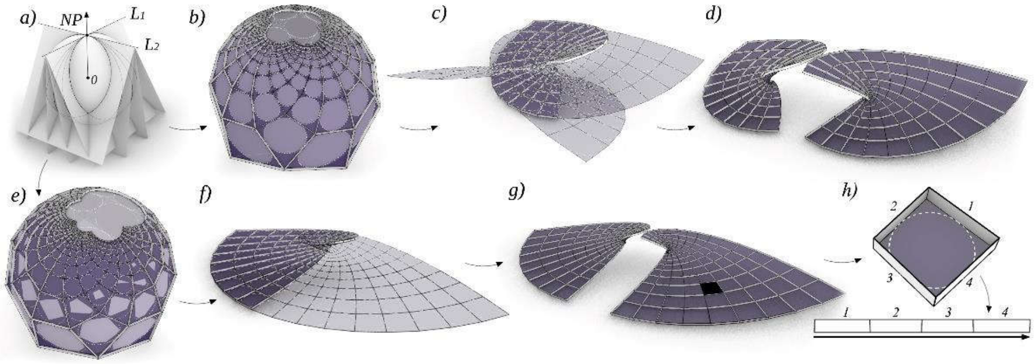

Figure 10. Level I: Construction via orthogonal plane strands and Christoffel duality for discrete minimal surfaces.

The last example involves constructing the conformal parametrization of the sphere using inverse stereographic projection. A geometric intuition of the mapping No (inverse stereographic projection) is as follows. It is network (on the sphere) can be obtained as the intersections of two sets of planes passing through orthogonal lines L_1 and L_2 at the north pole (Figure 10a). Once again, to obtain circle packing, we divide the u, v domain into squares upon which we have the initial standard circle packing, which is then sent by No to a circle packing on the sphere. Following the process, the Koebe mesh is constructed, and Christoffel duality is applied to form a discrete minimal surface (level I). If self-intersection occurs, an asymmetric section is extracted and adjusted symmetrically. A second variant is created by changing the domain parameters from the first GH component. To confirm that all surfaces derived from Christoffel duality are discrete minimal surfaces, the properties are checked according to geometry theory (cf. Section 2.3).

3.2 Procedure for Constructing Isothermic Meshes – Level II

This subsection focuses on constructing IS-meshes (Level II). In this case, the IS-mesh retains all fabrication advantages from the previous level, except for straight unfolding of box edges, as it is no longer an EO-mesh. Planarity, near orthogonality, and square-like quad proportions still remain. These are obtained by applying Möbius transformation (MT) to Level I meshes. Doing that, they also retain the properties of s-IS mesh of the quads as the MT keeps the inscribed circles and spheres inhereted from discrete minimal surface. However, the saddle condition is no longer garantied. The first example is the MS-mesh of Enneper type (Level I), transformed using MT based on formulas from Section 2.1.1, with parameters $a = 0.5, b = 0, c = -0.4, d = 0$. The result is an s-Isothermic mesh with kept circular mesh which is not an MS-mesh (Figure 11b). Duality is verified by constructing panels via diagonals and checking their planarity, which is satisfied, as well as their cross-ratios condition (Figure 11b, right detail).

As mentioned, the isothermic (circular) mesh has constant vertex offset, when defining the shell thickness. This discrete normal at a vertex is defined as the avergage of the four face nomales of the adjacent quads meeting at this vertex. Planarity checks confirmed no geometric torsion, as expected for circular meshes. As mentioned above, this is an important fabrication advantage as these planar lateral faces are the basis for the beams in the grid-shell.

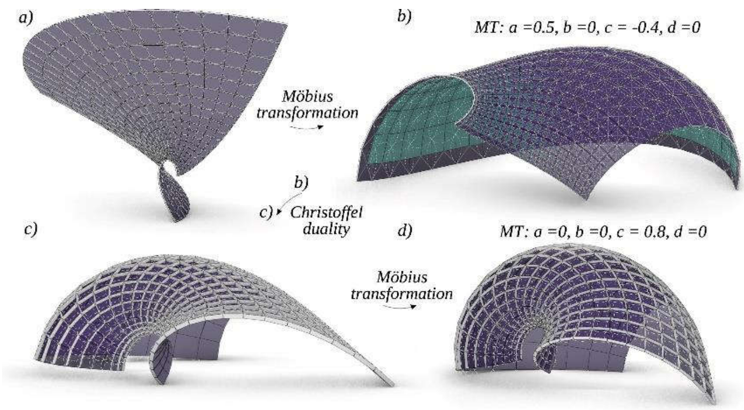

Figure 11. Level II Isothermic mesh variants from Enneper type MS-mesh via Möbius transformations and Christoffel duality.

To demonstrate the design tool, alternating Christoffel duality and Möbius transformation were applied, generating two more mesh variants (Figure 11c, d), with new MT parameters: $a = 0, b = 0, c = 0.8, d = 0$. A second iteration of the same principle was applied to a discrete minimal surface generated using a Möbius transform (in the plane), then applying the process described above (Figure 12a). To reach level II, a Möbius transformation was applied with parameters $a = 0.5, b = 0, c = -0.4, d = 0$ (Figure 12b). The resulting mesh retained circular properties, and Koenigs mesh properties were applied to generate (green) panels constructed along white diagonals that are also planar and therefore can be also fabricated as flat secondary covering (Figure 12b). The main (purple) panels are also planar, satisfy closure condition and therefore duality. Offset points were generated as before, confirming absence of geometric torsion.

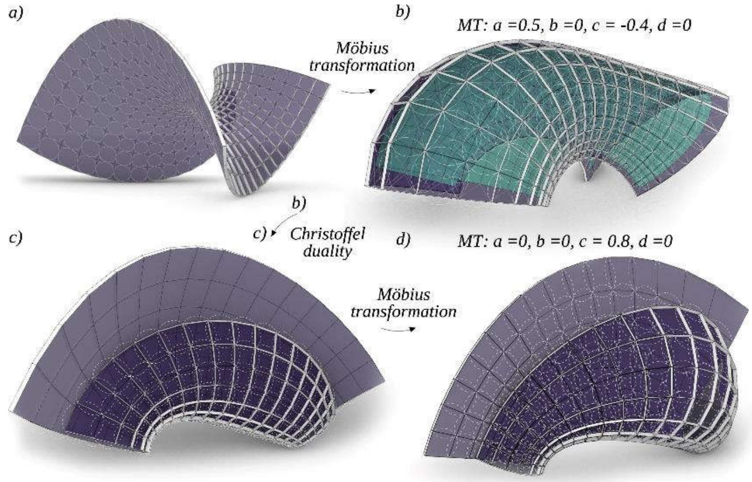

Figure 12. Level II Isothermic mesh variants from Christoffel duality and Möbius transformations.

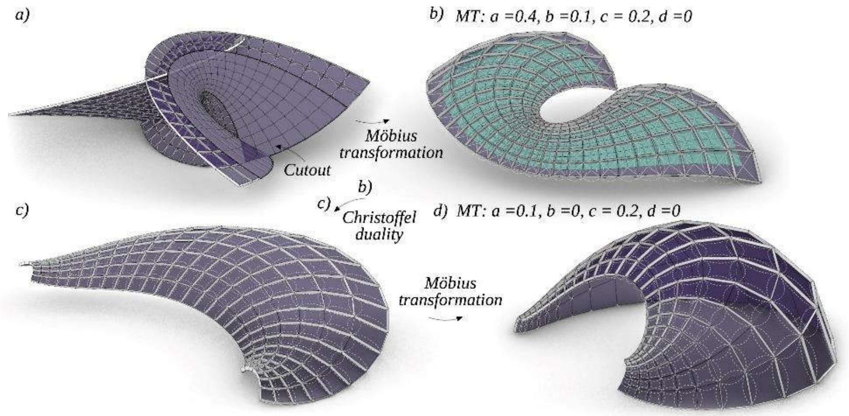

Figure 13. Level II Isothermic mesh variants from pencil of planes MS-mesh via Möbius transforms and Christoffel duality.

Two additional examples demonstrate alternating Christoffel duality (Figure 12c) and Möbius transformations ($a = 0, b = 0, c = 0.8, d = 0$, Figure 12d), showing how different isothermic mesh variants can be generated through iterative application. This process can be repeated for further exploration using the input parameters of the GH components and apply them successively.

The last case shows level I discrete minimal surfaces (from inverse stereographic projection) transformed into level II isothermic meshes (Figure 13a). The results are shown in Figure 13b–d. Parameter experiments revealed that: a and b increase asymmetry, c alters curvature, and d scales the grid-shell. The initial mesh position and size also influences the transformation effect—placing the mesh at $(0,0,0)$ and scaling it properly is recommended. Like mentioned, separate Grasshopper components were created for Möbius transformations and Christoffel duality, allowing easy alternation and further shape exploration. Thanks to the developed approach and tools, no matter which mesh shape is obtained the fabrication properties of the quads are kept. However, to demonstrate the strategy's full potential, the next step is level III, defined purely by Koenigs meshes without additional properties.

3.3. Construction Method of Koenigs Meshes – Level III

Level III meshes, defined as Koenigs meshes, involve only the application of duality4 and lack properties of minimal or isothermic meshes. At level III, the mesh retains only quad planarity as a fabrication advanatge. Being the least constrained, they can be generated in various ways. Recall that level III contains all the other levels in it. More precisely, level I is contained in level II which is contained in level III. In other words, an MS-mesh is an IS-mesh and a K-mesh and any IS-mesh is also a K-mesh. But not the other way around.

Koenigs meshes are preserved by projective transformation (PT), which maintains planarity but not IS-meshes nor MS-meshes. PT is applied by putting their formula presented above in Grasshopper as a component that recalculates point positions. Like with Möbius transformations, PT can alternate with duality to generate diverse mesh forms. Note that, since PT preserve planarity, the application of PT to IS-mesh including their planar later face (obtained by their normals) will result in K-meshes with later faces that are planar. However, the lines of intersections of these tranformed later faces are no longer functioning as vertex normals (recall K-meshes are not circular). This presents one additional fabrication advantage of also fabricating boxes from PQ strips (but not straight like in MS-mesh).

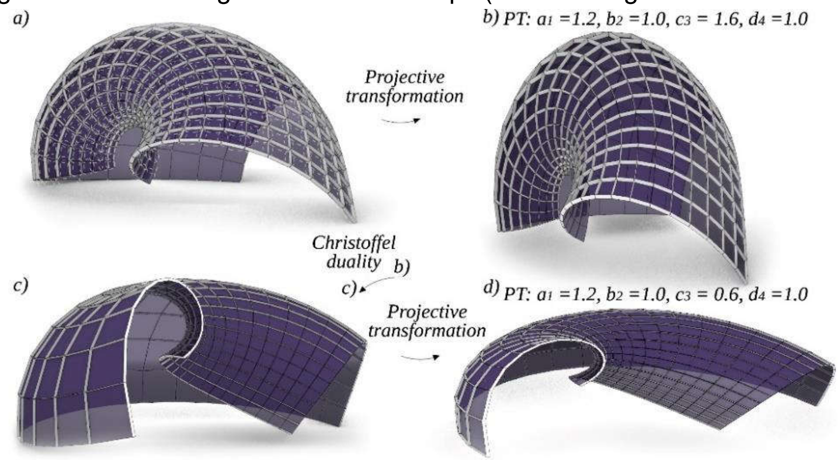

Figure 14. Koenigs mesh Level III variants via projective transformations and Christoffel duality from isothermic mesh from Figure 11d

The first example begins with a Level II mesh created by alternating Möbius transformation and Christoffel duality. A projective transformation is applied ($a_1 = 1.2, b_2 = 1.0, c_3 = 1.6, d_4 = 1.0$) resulting in a Level III mesh. When parameters a_1, b_2, c_3, and d_4 equal 1.0 (others are 0), the mesh remains unchanged—serving as a starting point for gradual adjustments to observe shape changes. Applying duality afterwards yields a different mesh shape, while a second PT further deforms it, producing another Level III variant. Three more Koenigs mesh examples are constructed similarly. Initial meshes were chosen from prior

[4] Note that we refer to it as Christoffel duality for IS and MS meshes, but simply as duality for Koenigs meshes—following smooth geometry conventions, even though the discrete construction is identical – same GH component.

example types. Starting from either Level I or II, PT and duality is applied iteratively using GH components and its input parameters to produce various Level III mesh forms.

In each example, the planar condition of panels and absence of geometric torsion were checked as well as planarity of the secondary mesh constructed by intersection of diagonals, which are both crucial fabrication benefits of Koenigs meshes. While planarity is naturally ensured by the projective transformation (PT). It is crucial to treat the initial mesh and its offset carefully when applying PT to an isothermic mesh, as was explained in the previous paragraph, namely the PT sends two (vertex) offsets of IS-meshes to two K-meshes that are not themselves offsets in the strict term (as the circularity is lost). Now, applying further a Chrsitoffel dual transform will further exacerbate the situation as the planarity of these later faces will be lost as well. To summarize, in dealing with purely K-meshes the notion of an offset mesh is not as clear as it is in IS-meshes (with their notion of Vertex Offset) or the MS-meshes (with their notion of Edge Offset). Thus, for K-meshes, we use the discrete normal at a vertex as the avergage of the four face nomales of the adjacent quads meeting at this vertex, checking if the resulting faces have no geometric torsion for the given offset, which turned out to be the case in all showcased examples.

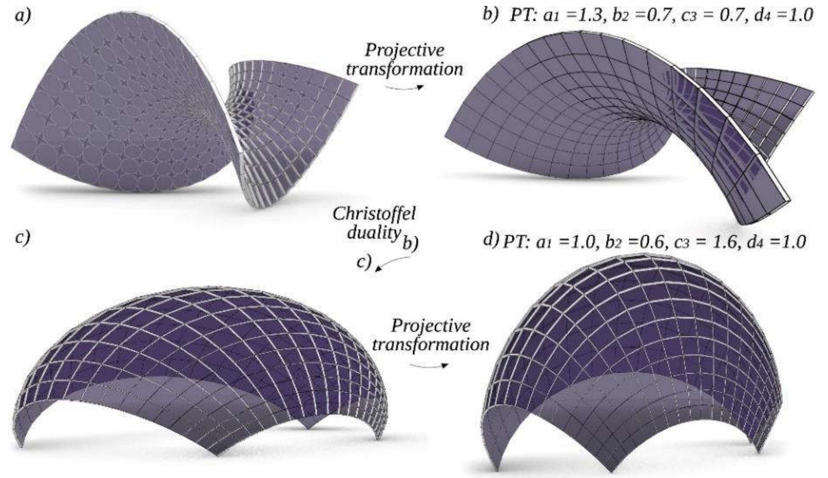

Figure 15. Koenigs mesh Level III variants from discrete minimal surface via projective transformations and Christoffel duality from mesh at Figure 12a

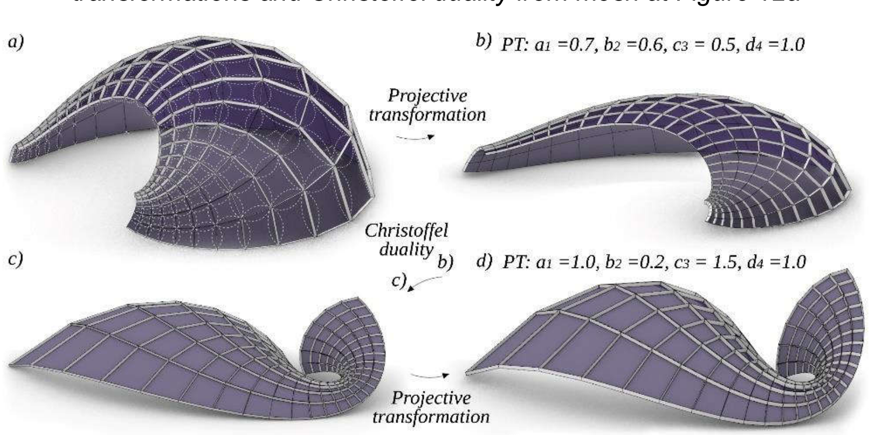

Figure 16. Koenigs mesh Level III variants from isothermic mesh via from Figure 13d projective transformations and Christoffel duality.

Experimenting with various PT input parameters revealed its effects differ significantly from those of the Möbius transformation. PT is more intuitive, as it involves affine transformations such as translation, scaling, rotation, reflection, and their combinations. Analyzing the examples, we found that parameters a and b stretch the mesh along the x and y axes, c stretches along the z-axis, and d scales the mesh. Like Möbius transformation, translating the global mesh in space will have a strong effect on the deformation of the outcome transformed mesh. In general, to better understand the morphological effect of the parameters it is advised to first keep the mesh in its standard position with respect to the origin before applying the transformation.

4. CONCLUSION

In this paper construction methods for Koenigs meshes in modular grid-shell design, were studied. Design strategies for discrete isothermic surfaces were formulated, a topic not extensively explored in architectural context. The same is done for the discrete minimal surfaces which possess special edge offset with straight unfolding quad blocks and square-like element proportions, favorable for lots of fabrication techniques.

Tools for designing these meshes were developed, categorized, and presented across three levels based on their fabrication constraints. Möbius and Projective transformations were then applied to preserve key geometric properties within each level while enabling limitless spatial variations. By following simple geometric rules, the developed methods and tools from different levels (I, II and III) can be used together by combining the developed GH components and trying out different input parameters. This flexibility increases design freedom and solutions within controlled local geometry of module which is one of main advantages of the developed strategy. The distinction between levels is crucial for the architectural implementation of the tools, as it should directly inform design decisions based on the available fabrication methods, as explained at the beginning of each subsection in Section 3. Also, developed tools help visualize the benefits and enable the intuitive implementation of fabrication-oriented design within pre-rationalization approach. They educate designers about these special mesh types that would otherwise be inaccessible and hard to implement. Algorithms were written in Grasshopper (GH) using a combination of available components and Python scripts with GH Python Remote. As separate components, these tools can be easily combined with each other and integrated with other Grasshopper components. This flexibility is crucial for adopting the framework across diverse architectural projects. Project-specific parameters—such as mesh orientation, scale, position, and element size or shape—can be defined by the designer and manipulated using the presented input parameters to suit the specific needs of each project.

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