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Research paper

ANALYSIS OF THE NATURAL FREQUENCY OF COMPOSITE PLATES MADE OF CROSS-LAMINATED TIMBER AND CONCRETE

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Abstract

This paper investigates the natural frequencies of square and rectangular cross-laminated timber–concrete composite plates using the approximate analytical Rayleigh method and finite element analysis in Abaqus/Standard. Determining the natural frequency, as dynamic characteristic, is important for verifying the serviceability limit state regarding the vibration criteria of such composite plates. The Rayleigh method, as an approximate method, allows for quick result acquisition, while modeling in Abaqus provides more detailed and accurate information about the plates dynamic characteristics. By comparing the results of both approaches, this paper examines the advantages and disadvantages of each method, offering guidelines for practical engineering applications. The conclusion of this study is that the Rayleigh method provides results for plates of characteristic dimensions that are in good agreement with those obtained using the Abaqus/Standard software package, which was used as a benchmark.

Key words: *composite plates, cross-laminated timber, concrete, natural frequency, Rayleigh method, finite element method*

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1. INTRODUCTION

For the design of floor plates, it is necessary to verify both the ultimate limit state and the serviceability limit state. Within the serviceability limit state, the vibration criterion, which is particularly relevant for composite plates, requires the determination of the natural frequency. According to the current Eurocode 5 regulations, in order to avoid more detailed dynamic analyses, the natural frequency of vibration of residential building timber floor systems must exceed 8 Hz [1]. In the case of composite timber-concrete floors, which represent an attractive alternative to conventional reinforced concrete plates due to their favorable properties, there is no explicit limit value for the natural frequency [2]. However, the same recommendations as for timber floors can be adopted. This is because a frequency below 8 Hz can cause discomfort to occupants and potentially coincide with the walking frequency of residents, which could lead to resonance effect and compromise the stability of the floor system.

Reinforced concrete plates have larger stiffness and mass, and typically exhibit higher natural frequencies compared to timber–concrete composite plates, making this condition generally insignificant. However, in the case of timber composite floor systems, which are relatively lightweight, the serviceability limit state with respect to vibrations may become the governing criterion for design.

Various models have been developed to determine the dynamic behavior of structures. Depending on the approach, these models may include analytical, numerical, or semi-numerical methods, which can be either exact or approximate. The most well-known among them are the Rayleigh method, Rayleigh-Ritz method, Galerkin method, Frobenius method, differential quadrature method, differential transformation method, finite difference method, and many others [3].

This paper analyzes the vibration frequencies of square and rectangular composite plates made of cross-laminated timber and concrete, obtained using two approaches: the Rayleigh method and finite element modeling in the software package "Abaqus" [4].

For the approximate analysis of the plate, the Kirchhoff plate theory (Classical Plate Theory, CPT) is used, which assumes that plane sections normal to the mid-surface before deformation remain plane and normal after deformation and neglects transverse shear deformation.

In this study an ideal bonding condition between the layers of the composite plate is imposed: perfect adhesion is assumed and no slip occurs at the interfaces. This idealisation simplifies the model and permits direct application of the Rayleigh method to estimate the natural frequencies. It should be noted that this assumption represents an upper bound on the structural stiffness and therefore may slightly overestimate the natural frequencies compared to real assemblies where some interlayer slip occurs. Experimental investigations have shown that interface behaviour and connector stiffness influence the effective bending stiffness and can shift the natural frequencies of timber–concrete composite plates and beams [5, 6]. In particular, differences between fully composite and partially composite beams can reach about 10% in natural frequency [5].

The results obtained using both methods are compared, and conclusions are drawn regarding the validity of the Rayleigh method for calculating the vibration frequency of timber-concrete composite plates.

2. GEOMETRIC AND MATERIAL PROPERTIES OF THE ANALYZED COMPOSITE PLATE

The analyzed plate consists of five layers of cross-laminated timber, each 3 cm thick, with a concrete plate of 8 cm thickness placed above them, resulting in a total plate thickness of 23 cm as shown in the Figure 1. Square plates with side lengths ranging from 4 m to 8 m in 1 m increments, and rectangular plates with a fixed span of 3 m and widths varying from 4 m to 8 m, were analyzed. All plates were simply supported along all four edges to investigate the influence of side length and width on their natural frequencies.

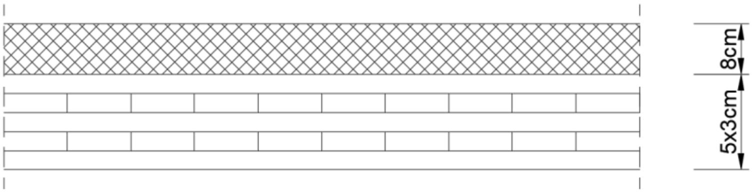

Figure 1. Cross section of composite plate

The timber used for the laminates is of grade C24, and the corresponding mechanical properties are provided in the Table 1 [7].

Table 1. Mechanical Properties of C24 Grade Timber

E_1 [MPa]	E_2 [MPa]	G_{12} [MPa]	G_{23} [MPa]	v_{12}	v_{21}	ρ [kg/m]
11000	370	690	50	0.38	0.01278	420

In the Table 1, E_1 and E_2 represent the Young's modulus in the principal directions, G_{12} and G_{23} are the shear modulus v_{12} and v_{21} are the Poisson's ratios, while ρ represents the material density.

The concrete used is of grade C25/30, and the corresponding mechanical properties are provided in Table 2.

Table 2. Mechanical Properties of C25/30 concrete

E [MPa]	G [MPa]	v	ρ [kg/m]
31000	1292	0.20	2400

3. RAYLEIGH'S METHOD

Rayleigh's method is an approximate analytical method used in structural dynamics to determine the natural frequencies of a system. It is based on the law of conservation of energy, which states that the sum of kinetic and potential energy remains constant. At the natural frequency, the ratio of maximum kinetic to potential energy is constant as follows:

$$\frac{T_{max}}{U_{max}} = \text{const} \tag{1}$$

where U_{max} represents maximum potential energy and T_{max} maximum kinetic energy.

For the composite plate, the maximum potential energy, depending on the transverse deflection of the plate w and the bending stiffness, D_{ij}, is defined as [8]:

$$U_{max} = \frac{1}{2}\iint \left(D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_{66}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \right) dxdy \quad (2)$$

while the maximum kinetic energy is represented in the form:

$$T_{max} = \frac{1}{2}\rho h \omega^2 \iint w^2 dxdy \quad (3)$$

In the previous equation ω represents the natural frequency, while h denotes the total thickness of the composite plate.

In laminate theory, D_{ij} represents the bending stiffness matrix and can be calculated as:

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{n}\left[\overline{Q}_{ij}\right]_k \left(h_k^3 - h_{k-1}^3\right) \quad (4)$$

where h_k represents the distance between the geometric midpoint of the composite plate and the edge of each layer, as shown in Figure 2.

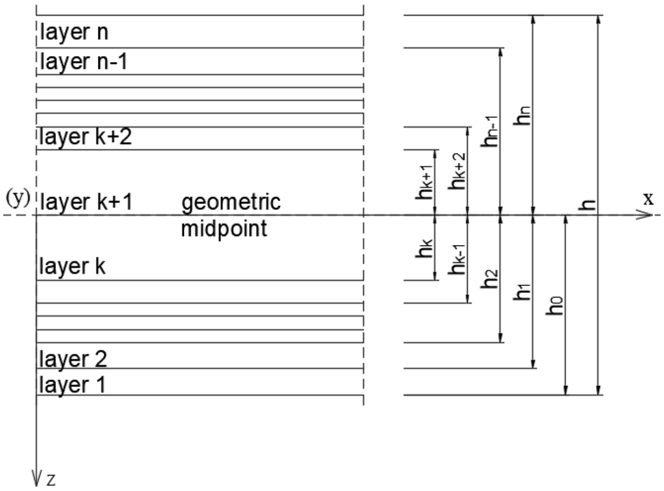

Figure 2. Layer designations in the composite

We define $\bar{z}_k$ as:

$$\frac{(h_k + h_{k+1})}{2} = \bar{z}_k \quad (5)$$

Thus, the bending stiffness matrix is now given as [8]:

$$D_{ij} = \sum_{k=1}^{n}\left[\overline{Q}_{ij}\right]_k \left(\frac{t_k^3}{12} + t_k \cdot \bar{z}_k^2\right) \quad (6)$$

where t_k represents thickness of k-th layer.

The matrix $\overline{Q_{ij}}$ is the reduced stiffness matrix, obtained as:

$$\overline{Q_{ij}} = [T]^{-1}[Q][R][T][R]^{-1} \tag{7}$$

where $[T]$ represents the transformation matrix, $[R]$ is the Reuter's matrix and $[Q]$ is the stiffness matrix, and the matrices are given as:

$$[T] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \tag{8}$$

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{9}$$

$$[Q] = \begin{bmatrix} \frac{E_1}{(1-\upsilon_{12}\cdot\upsilon_{21})} & \frac{E_2\cdot\upsilon_{12}}{(1-\upsilon_{12}\cdot\upsilon_{21})} & 0 \\ \frac{E_1\cdot\upsilon_{12}}{(1-\upsilon_{12}\cdot\upsilon_{21})} & \frac{E_2}{(1-\upsilon_{12}\cdot\upsilon_{21})} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \tag{10}$$

The approximation of the transverse deflection function using Rayleigh method is given by [9]:

$$w(x,y) = A \cdot X(x) \cdot Y(y) \tag{11}$$

where X and Y are the admissible functions that form the functional basis. They must satisfy the boundary conditions along the plate's contour. A represents the amplitude coefficients that need to be determined.

For the simply supported edges the characteristic function X and Y that satisfy the boundary conditions can be chosen as:

$$\begin{aligned} X(x) &= \cosh\lambda\frac{x}{a} - \cos\lambda\frac{x}{a} - \gamma(\sinh\lambda\frac{x}{a} - \sin\lambda\frac{x}{a}) \\ Y(y) &= \cosh\lambda\frac{y}{b} - \cos\lambda\frac{y}{b} - \gamma(\sinh\lambda\frac{y}{b} - \sin\lambda\frac{y}{b}) \end{aligned} \tag{12}$$

The coefficients λ (lambda) and γ (gamma) are determined using boundary conditions, requiring zero deflection along all edges of the plate and zero moment, where a and b represent the dimensions of the plate. [10]

In order to obtain the natural frequency one can apply the law of conservation of energy [9]:

$$\frac{\partial U}{\partial A} - \frac{\omega^2 \rho h}{2}\frac{\partial}{\partial A}\iint w^2 dxdy = 0 \tag{13}$$

For the analysis using the presented approximate analytical method, a MATLAB program code was developed, and the results are shown in Table 2 and Table 3.

Table 2. Natural frequencies of square plates vibrations obtained using the Rayleigh method

Dimensions of a square plate [m]	lambda (λ)	gamma (ϒ)	f (Hz)
4x4	1.676E-04	1.239E+00	28.48
5x5	1.661E-04	1.219E+00	18.22
6x6	1.655E-04	1.211E+00	12.66
7x7	1.652E-04	1.208E+00	9.30
8x8	1.651E-04	1.206E+00	7.12

Table 3. Natural frequencies of rectangular plates vibrations obtained using the Rayleigh method

Dimensions of a rectangular plate [m]	lambda (λ)	gamma (ϒ)	f (Hz)
3x4	1.701E-04	1.277E+00	39.40
3x5	1.694E-04	1.267E+00	33.81
3x6	1.692E-04	1.263E+00	30.60
3x7	1.691E-04	1.261E+00	28.58
3x8	1.691E-04	1.261E+00	27.23

4. MODELING IN ABAQUS SOFTWARE

To compare the results obtained using the approximate Rayleigh method, a plate model was created in the „Abaqus" software package [4] and material properties specified for each layer were defined using the "Lamina" option. The plate was modeled as a 2D composite with six layers, each with its corresponding material properties. Along the plate edges, pinned boundary conditions were applied: displacements in all three translational directions (U1, U2, U3) were restrained, while rotations were left free (Figure 3).

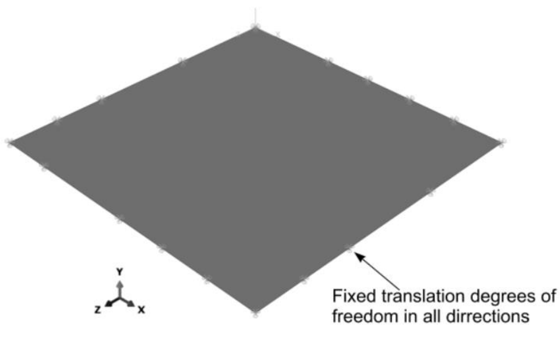

Figure 3. Plate model

The model is meshed using linear quadrilateral shell elements with four nodes, linear interpolation and reduced integration (S4R). A mesh convergence study was performed by varying the global seed size from 0.05 m to 0.15 m. The resulting values of natural frequencies exhibited a minimal variation of only 0.22 %, proving that the model yields reliable results. Therefore, a global seed size of 0.15 m was adopted in this study to facilitate more efficient computation.

The shape of the first natural mode for square plate 4x4m is shown in Figure 4. The corresponding mode shapes for rest square and rectangular plates exhibits a similar pattern and is therefore omitted for brevity. The vibration frequencies obtained for square plates of various dimensions are presented in Table 4, while those for rectangular plates are given in Table 5.

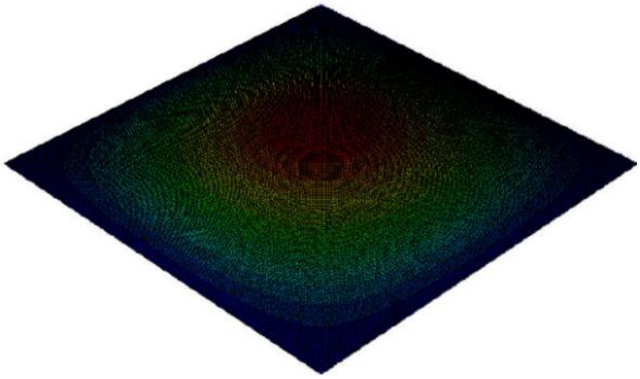

Figure 4. First natural mode of square plate 4x4m

Table 4. Natural frequencies obtained in Abaqus

Dimensions of a square plate [m]	f (Hz)
4x4	30.15
5x5	20.59
6x6	14.66
7x7	11.04
8x8	8.59

Table 5. Natural frequencies obtained in Abaqus

Dimensions of a rectangular plate [m]	f (Hz)
3x4	39.39
3x5	35.03
3x6	32.94
3x7	31.82
3x8	31.16

5. RESULTS DISCUSSION

The obtained results of frequencies for square and rectangular plates simply supported on all four edges were compared between the aproximate analytical analysis using the Rayleigh method of energy conservation in MATLAB and the finite element method using Abaqus software [4].

The data analysis shows that the frequencies obtained using the Rayleigh method are slightly lower compared to the values obtained in Abaqus. The difference is more pronounced for larger plate dimensions, while the deviations decrease as the dimensions decrease as shown in Table 6 and Table 7.

Table 6. Comparison of the obtained results for square plates

Dimensions of a square plates [m]	Rayleigh f [Hz]	Abaqus f [Hz]	Deviations [%]
4x4	28.48	30.15	5.54
5x5	18.22	20.59	11.51
6x6	12.66	14.66	13.64
7x7	9.30	11.04	15.76
8x8	7.12	8.59	17.11

Table 7. Comparison of the obtained results for rectangular plates

Dimensions of a rectangular plates [m]	Rayleigh f [Hz]	Abaqus f [Hz]	Deviations [%]
3x4	39.40	39.39	0.02
3x5	33.81	35.03	3.61
3x6	30.60	32.94	7.65
3x7	28.58	31.82	11.34
3x8	27.23	31.16	14.43

These differences can be attributed to the assumptions used in the Rayleigh method, which involve approximations in the expressions for potential and kinetic energy and the differences are graphically shown in Figure 5. and Figure 6.

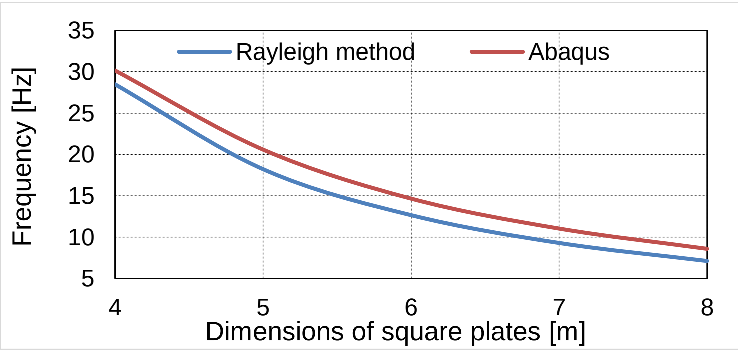

Figure 5. Comparative results of the dependence between natural frequency and plate dimensions for square plates

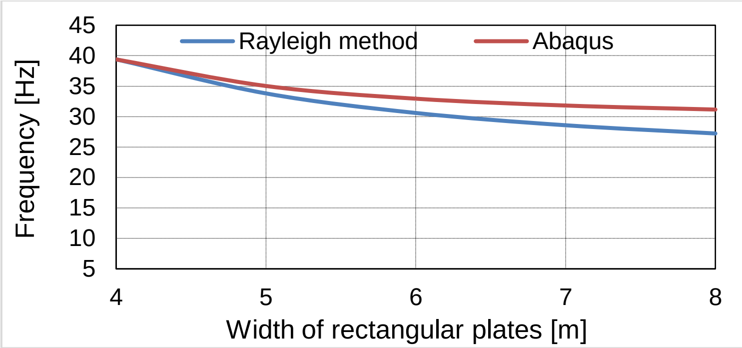

Figure 6. Comparative results of the dependence between natural frequency width of rectangular plates (constant span of 3 m)

In general, it can be concluded that the analytical (Rayleigh) method shows good agreement with the finite element analysis results for plates with characteristic dimensions. Both approaches exhibit the same trend of frequency reduction with increasing plate size, confirming the consistency of the models. However, due to the more conservative kinematic description of the plate in the Rayleigh method, the calculated natural frequencies are slightly lower than those obtained with the Abaqus software.

6. CONCLUSION

This paper presents two methods for determining the natural frequency of composite plates made of laminated timber and concrete using the Rayleigh method and modeling in Abaqus software. Based on the presented results, it can be concluded that the methods are in good agreement, with small deviations observed for plates with shorter spans and increasing deviations as the plate dimensions become larger. The results indicate that the natural frequency of composite plates can be reliably calculated using the Rayleigh method, making it suitable for use in engineering practice.

In addition to the assumption of ideal bonding between timber and concrete, which affects the results, only a single term was used in the Rayleigh method, which limited the accuracy in predicting the natural frequencies. To further improve the accuracy and expand the validity of the results, it would be beneficial to implement an extended model with two or more terms in the deflection function approximation, which would represent the Rayleigh-Ritz method. This approach would allow for a more detailed analysis of the plate vibrations and could lead to more accurate results, especially in more complex cases.

Additionally, in this study, only the self-weight of the composite plate was considered as the load, which represents a simplified approach. To more accurately simulate actual conditions, it would be beneficial to include an additional dead load that influences the plate's vibrations. This can be achieved by increasing the material density in the analytical Rayleigh method.

Additionally, it could be considered to expand the research to include different types of plates (varying layer thicknesses, or different materials) in order to analyze the generalizability of the methods in a broader context. Also selection of different trial functions for X and Y could represent a further step toward a more accurate application of the Rayleigh method for analyzing the behavior of plates under various conditions.

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